Percolation in a Fractional Brownian Motion Lattice

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Many applications of engineering interest involve spatially correlated properties obeying the statistics of fractional Brownian motion (fBm) (Mandelbrot and van Ness, 1968). Because of its spatially growing correlations (Feder, 1988), fBm is a special, but important example of spatial correlation. In applications related to chemical engineering, fBm statistics have been shown to describe: pressure fluctuations in a bubble column (Drahos et al., 1992) or solid-liquid fluidized beds (Fan et al., 1993; Kwon et al., 1994); the surface morphology of fracture surfaces, such as poly(styrene) and poly(methyl methacrylate) (Talibuddin and Runt, 1994) and more generally of rough substrates (Gunter and Niemantsverdriet, 1995); the pore surface of various porous media (Mann, 1993); and the trace of propagating cracks (Schmittbuhl et al., 1994). Fractional Brownian motion was also conjectured to describe the distribution of permeabilities in various oil reservoirs (Hewett, 1986) in an attempt to explain field data of Arya et al. (1988), which indicated spatially growing correlations. In fact, the so-called fractal geostatistics have been successfully used in various practical situations (e.g., Emmanuel et al., 1989).

Of specific interest to this note are percolation processes in a field described by fBm. There are two applications where such processes may arise: The immiscible displacement of one fluid by another in a 2-D fracture, the aperture of which obeys fBm statistics; and the same displacement in a heterogeneous porous medium, the permeability of which obeys fBm statistics. In either case, it is assumed that capillary forces dominate the displacement. In previous works, Yortsos and Chang (1990) and van Batenburg et al. (1991) simulated displacements in such fields by considering both capillary and viscous forces. Here, we consider the case where viscous forces are small (low flow rates). As capillary-controlled displacement is described by invasion percolation (IP) (Wilkinson and Willemsen, 1983, see also recent upscaling to random permeabilities by Yortsos et al., 1993), the problem of interest is one of percolation in a correlated lattice.

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Depending on the behavior of the correlation function at large values of the spatial lag, three cases of site percolation in a correlated lattice have been studied:

(i) When the decay is exponential, Renault (1991) showed that the percolation threshold p_c of a correlated lattice is smaller than that of a random lattice (see also Ioannidis and Chatzis, 1993).

(ii) When the correlation function decays algebraically, such as following $C(\rho) \sim \rho^{2H}$, $H<0$, Prakash et al. (1992) found that in site percolation in a square lattice p_c decreases with increasing H, and approaches the value of 0.5 at $H=0$. For this problem, Isichenko (1992) showed that the percolation exponent β, which scales the behavior of the percolation probability near p_c, remains the same as in the uncorrelated case. However, the correlation length exponent ν changes according to $\nu = -1/H$, $-0.75 < H < 0$. Using the relationship $D_f = d - \beta/\nu$, these results show that the fractal dimension D_f of the percolation cluster increases with H ($H<0$), and approaches the embedding dimension $d=2$ as $H \to 0$.

(iii) When the correlation function grows algebraically, as in an fBm lattice, where $H>0$, Satik and Yortsos (1991) reported in an unpublished study of IP that both the percolation threshold p_c and the percolation probability function $P(p)$ are stochastic, different realizations resulting in different p_c values and different $P(p)$ curves. In particular, these curves did not converge to a universal curve with an increase in the lattice size, as is the case with ordinary percolation (OP) in a random lattice. A theoretical understanding of this behavior was provided by Isichenko (1992), who analyzed 2-D OP in fBm lattices and proposed that the mass fractal dimension D_f is constant and equal to 2 (a compact cluster), although that of its perimeter, D_h, varies with H as $D_h = (10-3H)/7$. Schmittbuhl et al. (1993) studied percolation in an fBm hierarchical lattice and showed analytically that the percolation threshold is indeed a stochastic variable with a finite variance. Sahimi (1994, 1995) reported deterministic percolation thresholds, which increase with H for $0<H<0.5$ and decrease for $0.5<H<1$, and fractal percolation clusters with a fractal dimension equal to that of the random case, except for H near 1. In this note, we present results which confirm our previous findings (Satik and Yortsos, 1991) and is consis-

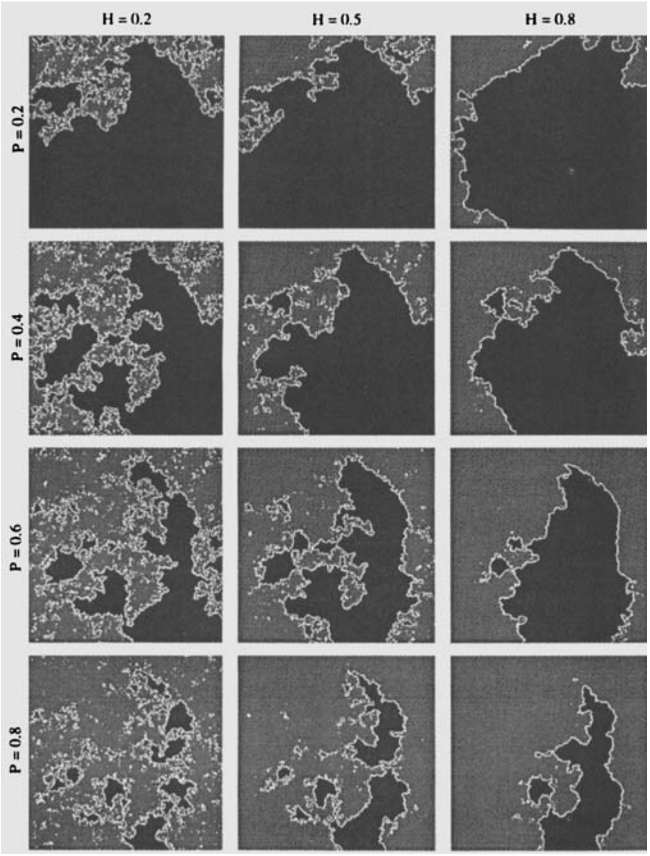

Figure 1. Consecutive snapshots of invasion at different values of the Hurst exponent H (0.2, 0.5, and 0.8) and invaded fraction P (0.2, 0.4, 0.6, 0.8). Dark color denotes defending phase.

tent with the theory of Isichenko (1992) and Schmittbuhl et al. (1993). Certain average properties are also computed.

We consider site IP in an fBm 2-D square lattice with sizes as large as 256×256. Sizes were assigned to the sites following fBm statistics, using the method of midpoint displacement and random addition described by Voss (1988) (see also Feder, 1988). An invasion process (from top-to-bottom in Figure 1) was then simulated, using the standard rule that the front advances to the largest perimeter site, and applying no-flow conditions at the lateral sides. Snapshots of the displacement for different values of H are shown in Figure 1. As in Satik and Yortsos (1991), the invaded regions appear to be compact, but with a fractal perimeter. Contrary to the random case, the features of the invading phase depend critically on the realization, which here can be simply changed by reversing the displacement (such as from bottom-to-top, from left-to-right, and so on), as can also be seen from a cursory inspection of the fBm field. From the data obtained, we calculated IP curves $P(p, H)$ for different realizations and H values. Here, $P(p, H)$ denotes the fraction of invaded sites, and as is conventional in IP, we defined $p = \int_{r_{\min}}^{\infty} \alpha(r)dr$, where $r_{\min}$ is the minimum size of invaded sites and $\alpha(r)$ is the probability density function (pdf).

Figure 2 shows various $P(p,H)$ curves for $H = 0.5$, corresponding to different realizations. The following are noted: (1) Contrary to the random case, the curves are stochastic,

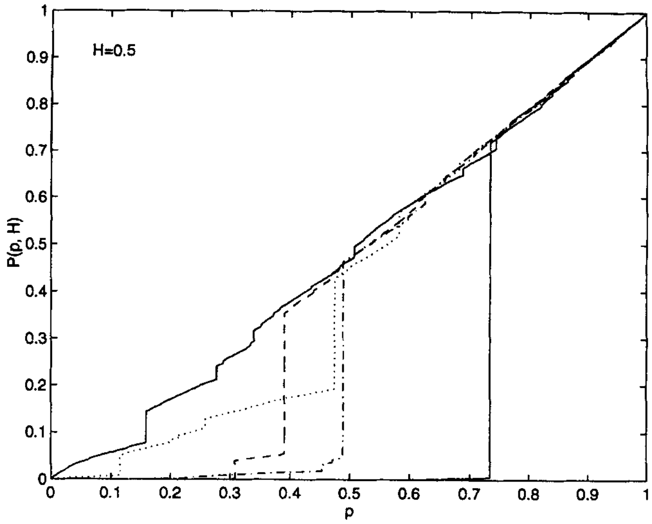

Figure 2. Invaded fraction curve $P(p, H)$ for $H = 0.5$ and different realizations.

different realizations for the same H giving rise to a different curve; (2) because the invasion cluster is not fractal (see also below), these curves assume nonzero values much before "breakthrough" of the invading phase; (3) for a given realization, each curve consists of a succession of jumps, which is characteristic of IP (Roux and Guyon, 1989). The frequency of the bursts decreases as H increases. These results are consistent with the earlier findings (Satik and Yortsos, 1991) and with the theory of Schmittbuhl et al. (1993). Certainly, universality in percolation properties does not apply in these systems, each invasion process being sensitive to the particular realization.

The percolation threshold p_c was studied by conducting a site OP process, using the cluster labeling algorithm of Hoshen and Kopelman (1976) (see also Stauffer and Aharony, 1992). It was found (Table 1) that p_c is a *random* variable, with a mean value $\langle p_c \rangle (H)$ that monotonically decreases with H, such that $\langle p_c \rangle (0) = 0.5$ (the value reached by Prakash et al., 1992 as H approaches zero from the negative side) and $\langle p_c \rangle (1) = 0.386$. Its standard deviation, $\sigma_p(H)$, is an increasing function of H with $\sigma_p(0) \approx 0$ (the nonzero value due to finite size effects) and $\sigma_p(1) \approx 0.13$. These are also consistent with Schmittbuhl et al. (1993).

Using these data, the density of the percolation cluster at p_c, $\rho(p_c, H) = P(p_c)/L^2$, was computed for various values of H and lattice sizes. The results, listed in Table 2, show that although stochastic, the variable $\rho(p_c, H)$ has mean and standard deviation practically independent of the lattice size, at least for large values of H. This should be contrasted to the random case, where the average density decreases with L, following the well-known power-law scaling $\rho \sim L^{-\beta/\nu}$, where $\beta = 5/36$ and $\nu = 4/3$. The lack of sensitivity to L (at least for larger H) confirms the observation that the percolation cluster is not fractal, in agreement with Isichenko's (1992) argument ($\nu \to \infty$). It must be noted that the slow decrease for $H = 0.2$, shown in Table 2, most likely reflects finite size effects, which should disappear at larger L. Values of the perimeter fractal dimension D_h were also estimated and found in reasonable agreement with Isichenko's (1992) prediction.

Table 1. Percolation Threshold p_c for Site Percolation in fBm Lattice

	Lattice Size/No. of Realizations				
H	64×64/1,000	128×128/500	256×256/100	512×512/100	1,024×1,024/100
1.00	0.3843±0.1307	0.3817±0.1274	0.3871±0.1295	0.3864±0.1291	0.3826±0.1263
0.80	0.3944±0.1247	0.3924±0.1218	0.3986±0.1240	0.3912±0.1261	0.3944±0.1196
0.65	0.4040±0.1206	0.4030±0.1182	0.4131±0.1212	0.4103±0.1212	0.4074±0.1181
0.50	0.4180±0.1164	0.4164±0.1133	0.4269±0.1153	0.4221±0.1132	0.4204±0.1137
0.35	0.4365±0.1086	0.4335±0.1060	0.4471±0.1061	0.4431±0.1030	0.4424±0.1057
0.20	0.4876±0.0802	0.4577±0.0958	0.4646±0.0931	0.4700±0.0882	0.4647±0.0875
0.00	0.4907±0.0782	0.4931±0.0740	0.4976±0.0670	0.5038±0.0673	0.5024±0.0607
Rand	0.5818±0.0216	0.5872±0.0121	0.5893±0.0077	0.5908±0.0044	0.5912±0.0028

Table 2. Density of Percolation Cluster at Breakthrough, $\rho = P(p_c)/L^2$, for Site Percolation in fBm Lattice

	Lattice Size/Number of Realizations				
H	64×64/1,000	128×128/550	256×256/550	512×512/200	1,024×1,024/100
0.80	0.3583±0.1263	0.3538±0.1269	0.3555±0.1277	0.3513±0.1241	0.3485±0.1099
0.50	0.3485±0.1133	0.3425±0.1136	0.3423±0.1166	0.3364±0.1120	0.3361±0.1071
0.20	0.3298±0.0985	0.3230±0.0998	0.3185±0.0999	0.3025±0.0901	0.3004±0.0838
Rand	0.2563±0.0748	0.2343±0.0684	0.2215±0.0674	0.2024±0.0590	0.1854±0.0548

Next, we computed ensemble averages $\mathcal{P}(p, H)$ by ensemble-averaging p for fixed P and H. The results are plotted in Figure 3 for different H values. The ensemble averages continuously deviate from the universal curve (random medium) as H increases, although there is some cross-over at large p. Lattice-size effects on the ensemble average curves were found to be negligible. As with p_c, the standard deviation was found to increase with H and to be larger as p decreases. A capillary pressure, $P_c \sim 1/r$, was also computed, where r is the size of the site currently being invaded. For the random case, the plot of the capillary pressure vs. time (or saturation S of the invading fluid) has the noisy appearance of Figure 4a, the properties of which were analyzed by Roux and Guyon (1989). For IP in an fBm lattice, curves corresponding to various H are shown in Figures 4b–4d. The following two characteristics are noted: (1) Contrary to the random case where it is flat, the envelope of P_c in the fBm case has a definite gradient, which increases with H; (2) the fluctuations diminish in size as H increases, reflecting the improved connectivity of the lattice, while the distribution of burst sizes is much narrower. The analysis of these bursts is currently under study.

In summary, in this note we presented numerical results of both OP and IP in an fBm lattice, which expand on our previous study (Satik and Yortsos, 1991) and support the findings of Isichenko (1992) and Schmittbuhl et al. (1993). The statistics of the random variables p_c and $P(p,H)$ obtained should be useful in problems involving fBm lattices. More generally, the results should be useful in the study of invasion processes in correlated media.

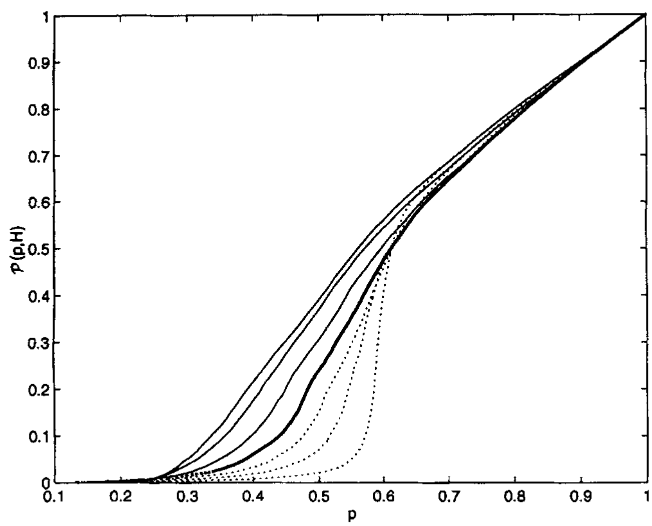

Figure 3. Ensemble average $\mathcal{P}(p, H)$ for different values of the Hurst exponent H (random, −0.5, −0.2, 0.0, 0.2, 0.5 and 1.0 from right to left).

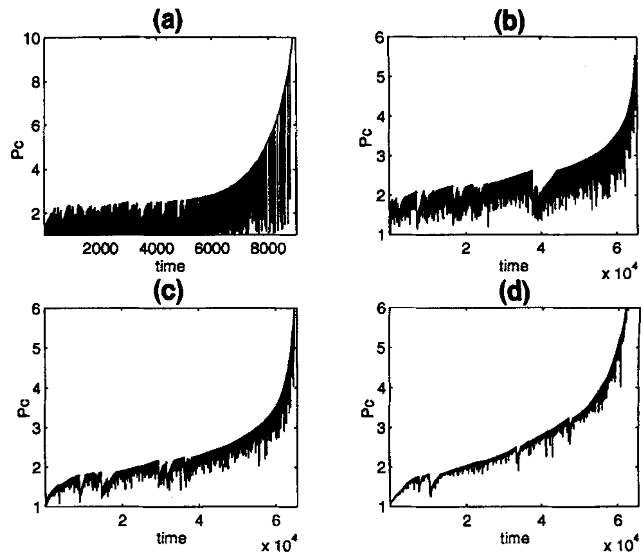

Figure 4. Normalized capillary pressure vs. invasion time for different values of the Hurst exponent H and a fixed realization.

(a) Random; (b) 0.2; (c) 0.5; (d) 0.8.

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Literature Cited

Arya, A., T. A. Hewett, R. G. Larson, and L. W. Lake, "Dispersion and Reservoir Heterogeneity," *SPERE*, **3**, 139 (1988).

Drahos, J., F. Bradka, and M. Puncochar, "Fractal Behavior of Pressure-Fluctuations in a Bubble Column," *Chem. Eng. Sci.*, **47**, 4069 (1992).

Emmanuel, A. S., G. K. Alameda, R. A. Behrens, and T. A. Hewett, "Reservoir Performance Prediction Methods Based on Fractal Geostatistics," *SPERE*, **4**, 311 (1989).

Fan, L. T., Y. Kang, D. Neogi, and M. Yashima, "Fractal Analysis of Fluidized Particle Behavior in Liquid-Solid Fluidized Beds," *AIChE J.*, **39**, 513 (1993).

Feder, J., *Fractals*, Plenum, New York (1988).

Gunter, P. L. J., and J. W. Niemantsverdriet, "Thickness Determination of Uniform Overlayers on Rough Substrates by Angle-Dependent XPS," *Appl. Surf. Sci.*, **89**, 69 (1995).

Hewett, T. A., "Fractal Distributions of Reservoir Heterogeneity and their Influence on Fluid Transport," SPE 15386, Technical Conf. and Exhib. of Soc. of Petrol. Engrs., New Orleans, LA (1986).

Hoshen, J., and R. Kopelman, "Percolation and Cluster Distribution: I. Cluster Multiple Labeling Technique and Critical Concentration Algorithm," *Phys. Rev. B*, **14**, 3428 (1976).

Ioannidis, M. A., and I. Chatzis, "The Effect of Spatial Correlations on the Accessibility Characteristics of Three-Dimensional Cubic Networks as Related to Drainage Displacements in Porous Media," *Wat. Res. Res.*, **29**, 1777 (1993).

Isichenko, M. B., "Percolation, Statistical Topography, and Transport in Random Media," *Rev. Mod. Phys.*, **64**, 961 (1992).

Kwon, H. W., Y. Kang, S. D. Kim, M. Yashima, and L. T. Fan, "Bubble-Chord Length and Pressure Fluctuations in Three-Phase Fluidized Beds," *Ind. Eng. Chem. Res.*, **33**, 1852 (1994).

Mann, R., "Developments in Chemical Reaction Engineering: Issues Relating to Particle Pore Structures and Porous Materials," *Chem. Eng. Res. & Des.*, **71**, A5, 551 (1993).

Mandelbrot, B. B., and J. W. Van Ness, "Fractional Brownian Motion, Fractional Noises and Application," *SIAM. Rev.*, **10**, 422 (1968).

Prakash, S., S. Havlin, M. Schwartz, and H. E. Stanley, "Structural and Dynamical Properties of Long-Range Correlated Percolation," *Phys. Rev. A*, **64**, R1724 (1992).

Renault, P., "The Effect of Spatially Correlated Blocking-Up of Some Bonds or Nodes of a Network on the Percolation Threshold," *Transp. Porous Media*, **6**, 451 (1991).

Roux, S., and E. Guyon, "Temporal Development of Invasion Percolation," *J. Phys. A: Math. Gen.*, **22**, 3693 (1989).

Sahimi, M., "Long-Range Correlated Percolation and Flow and Transport in Heterogeneous Porous Media," *J. Phys. I*, **4**, 1263 (1994).

Sahimi, M., "Effect of Long-Range Correlations on Transport Phenomena in Disordered Media," *AIChE J.*, **41**, 229 (1995).

Sahimi, M., "The Percolation Cluster Is Compact in 2-D and 3-D, Although Its Backbone Is Not," personal communication (1996).

Satik, C., and Y. C. Yortsos, "Simulation of Pattern Formation and Percolation Using Fractional Brownian Statistics," AIChE Meeting, Los Angeles (Nov. 17–22, 1991).

Schmittbuhl, J., S. Roux, and Y. Berthaud, "Development of Roughness in Crack Propagation," *Europhys. Lett.*, **28**, 585 (1994).

Schmittbuhl, J., J.-P. Vilotte, and S. Roux, "Percolation through Self-Affine Surface," *J. Phys. A: Math. Gen.*, **26**, 6115 (1993).

Stauffer, D., and A. Aharony, *Introduction to Percolation Theory*, Taylor and Francis, London and Washington (1992).

Talibuddin, S., and J. P. Runt, "Reliability Test of Popular Fractal Techniques Applied to Small Two-Dimensional Self-Affine Data Sets," *J. Appl. Phys.*, **76**, 5070 (1994).

van Batenburg, D. W., J. Bruining, H. Bakken, and C. T. S. Palmgren, "The Effect of Capillary Forces in Heterogeneous 'Flow-Units': a Streamline Approach," SPE 22588, Technical Conf. and Exhib. of Soc. of Petrol. Engrs., Dallas (1991).

Voss, R. F., *The Science of Fractal Images*, H.-O. Peitgen and D. Saupe, eds., Springer-Verlag, Berlin (1988).

Wilkinson, D., and J. F. Willemsen, "Invasion Percolation: a New Form of Percolation Theory," *J. Phys. A: Math. Gen.*, **16**, 3365 (1983).

Yortsos, Y. C., and J. Chang, "Capillary Effects on Steady State Flow in Heterogeneous Cores," *Transp. Porous Media*, **5**, 399 (1990).

Yortsos, Y. C., C. Satik, J.-C. Bacri, and D. Salin, "Large-Scale Percolation Theory of Drainage," *Transp. Porous Media*, **10**, 171 (1993).

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